In the beginning, there was *symmetric* encryption.

If you had the key you could encrypt ...

If you had the key you could encrypt ...

Message:	**ATTACK AT DAWN**
Key: +3	↓↓↓↓↓↓ ↓↓ ↓↓↓↓
Ciphertext:	**DWWDFN DW GDZQ**

If you had the key you could decrypt …

If you had the key you could decrypt …

Message:	**ATTACK AT DAWN**
Key: +3	↑↑↑↑↑↑ ↑↑ ↑↑↑↑
Ciphertext:	**DWWDFN DW GDZQ**

… and some people were happy.

Then, there was asymmetric encryption.

Some people encrypted …

Some people encrypted …

… others decrypted.

E-commerce ensued …

E-commerce ensued …

… and more people were happy.

The first and most used asymmetric cipher was RSA.

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$$E(m) = m^e \pmod{n}$$

Some people noticed the algebraic structure …

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$$E(m_1) = {m_1}^e \qquad E(m_2) = {m_2}^e$$

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Ergo …

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Ergo …

$$E(m_1) \times E(m_2)$$

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$$E(m_1) = m_1{}^e \qquad E(m_2) = m_2{}^e$$

Ergo …

$$E(m_1) \times E(m_2)$$

$$= m_1{}^e \times m_2{}^e$$

Some people noticed the algebraic structure …

$$E(m_1) = {m_1}^e \qquad E(m_2) = {m_2}^e$$

Ergo …

$$\begin{aligned} E(m_1) \times E(m_2) \\ &= {m_1}^e \times {m_2}^e \\ &= (m_1 \times m_2)^e \end{aligned}$$

Some people noticed the algebraic structure …

$$E(m_1) = m_1{}^e \qquad E(m_2) = m_2{}^e$$

Ergo …

$$\begin{aligned} & E(m_1) \times E(m_2) \\ & \qquad = m_1{}^e \times m_2{}^e \\ & \qquad = (m_1 \times m_2)^e \\ & \qquad = E(m_1 \times m_2) \end{aligned}$$

They looked for interesting applications …

They looked for interesting applications …

… and they failed.

People mused …

People mused …

… if only RSA worked additively …

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… if only RSA worked additively …

we could compute sums …

People mused …

… if only RSA worked additively …

we could compute sums …

and averages …

People mused …

… if only RSA worked additively …

we could compute sums …

and averages …

and tally elections …

I was one of those musing.

I was one of those musing.

An additive encryption homomorphism …

I was one of those musing.

An additive encryption homomorphism …

$$E(m, r) = r^e c^m$$

$$E(m_1, r_1) = {r_1}^e c^{m_1} \qquad E(m_2, r_2) = {r_2}^e c^{m_2}$$

$$E(m_1, r_1) = r_1{}^e c^{m_1} \qquad E(m_2, r_2) = r_2{}^e c^{m_2}$$

$$E(m_1, r_1) \times E(m_2, r_2)$$

$$E(m_1, r_1) = \ {r_1}^e c^{m_1} \qquad E(m_2, r_2) = \ {r_2}^e c^{m_2}$$

$$E(m_1, r_1) \times E(m_2, r_2)$$
$$= {r_1}^e c^{m_1} \times {r_2}^e c^{m_2}$$

$$E(m_1, r_1) = r_1{}^e c^{m_1} \qquad E(m_2, r_2) = r_2{}^e c^{m_2}$$

$$\begin{aligned} E(m_1, r_1) \times E(m_2, r_2) \\ &= r_1{}^e c^{m_1} \times r_2{}^e c^{m_2} \\ &= (r_1 r_2)^e c^{m_1+m_2} \end{aligned}$$

$$E(m_1, r_1) = \ {r_1}^e c^{m_1} \qquad E(m_2, r_2) = \ {r_2}^e c^{m_2}$$

$$\begin{aligned} &E(m_1, r_1) \times E(m_2, r_2) \\ &\quad = {r_1}^e c^{m_1} \times {r_2}^e c^{m_2} \\ &\quad = (r_1 r_2)^e c^{m_1 + m_2} \\ &\quad = E(m_1 + m_2, r_1 r_2) \end{aligned}$$

$$E(m_1, r_1) = r_1{}^e c^{m_1} \qquad E(m_2, r_2) = r_2{}^e c^{m_2}$$

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The product of encryptions of two messages is *an* encryption of the sum of the two messages.

I used this to build verifiable election systems …

I used this to build verifiable election systems …

… and I was *really* happy …

I used this to build verifiable election systems …

… and I was *really* happy …

and few others cared.

What people really wanted was the ability to do arbitrary computing on encrypted data…

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… and this required the ability to compute *both* sums *and* products …

What people really wanted was the ability to do arbitrary computing on encrypted data…

… and this required the ability to compute *both* sums *and* products …

… on the same data set!

People tried to do this for years …

People tried to do this for years …

… and years …

People tried to do this for years …

… and years …

… and years …

People tried to do this for years …

… and years …

… and years …

… with no success.


STAR WARS
RETURN OF THE JEDI


STAR WARS
THE CLONE WARS
IN THEATERS AUGUST 15

WHY does ADD AND MULTIPLY help?

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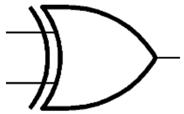
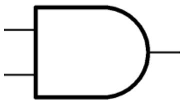


XOR (add mod 2)

0 XOR 0	0
1 XOR 0	1
0 XOR 1	1
1 XOR 1	0



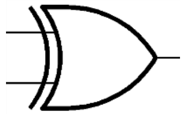
AND (mult mod 2)

0 AND 0	0
1 AND 0	0
0 AND 1	0
1 AND 1	1

WHY does ADD AND MULTIPLY help?

... because {XOR,AND} is Turing-complete ...

(any function can be written as a combination of XOR and AND gates)



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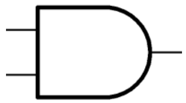

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WHY does ADD AND MULTIPLY help?

... because {XOR,AND} is Turing-complete ...

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Example: Searching a database

DB

0
1
1
0

index

$I = i_1 i_0$

$\longleftarrow$

answer= DB_I

$\longrightarrow$

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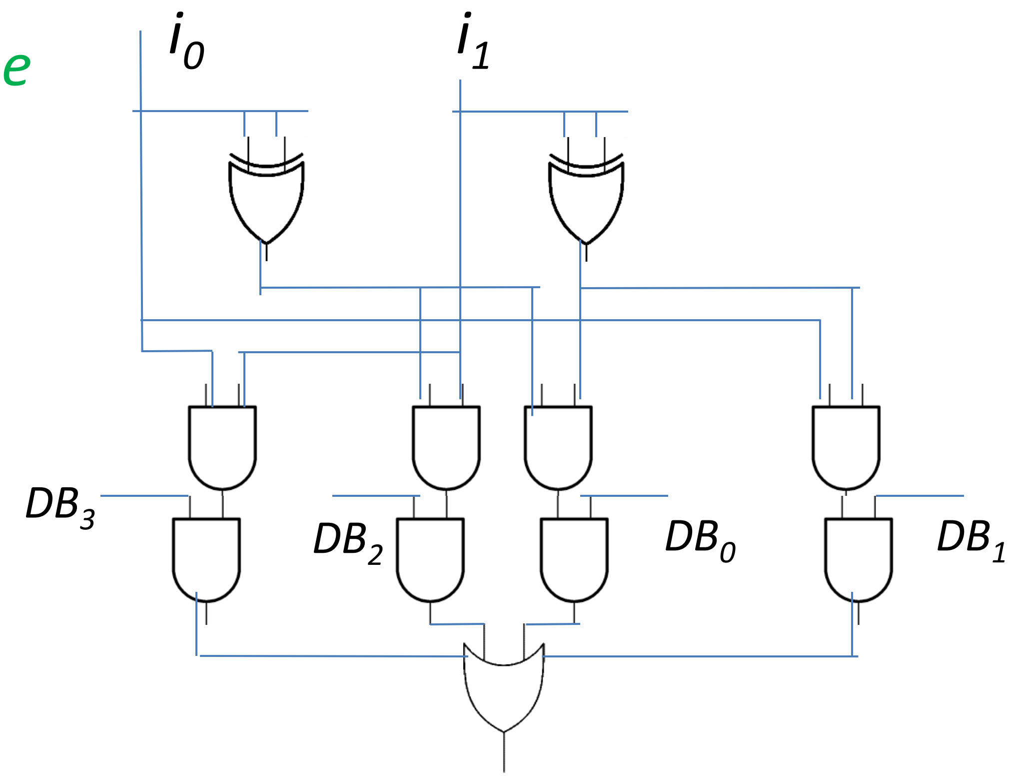
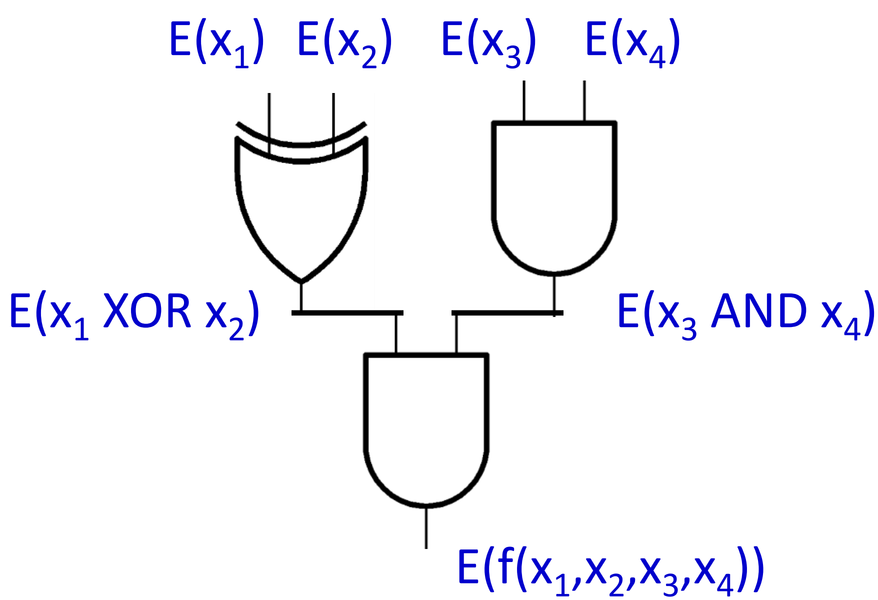


WHY does ADD AND MULTIPLY help?

… because {XOR,AND} is Turing-complete …

… if you can compute XOR and AND on encrypted bits…

… you can compute ***ANY*** function on encrypted inputs…



This is A M A Z I N G!

Private bing Search

Private Cloud computing

This is A M A Z I N G!

Private bing Search

Private Cloud computing

In general,

Delegate *processing* of data

without giving away *access* to it

People tried to compute both AND and XOR on encrypted bits …

… for years …

… and years …

… with no success.

Well, actually, there were some ***partial*** answers …

Fully homomorphic

Josh's system

MANY add

ZERO mult

MANY add

MANY mult

Well, actually, there were some ***partial*** answers …

Fully homomorphic

Josh's system

Boneh,Goh & Nissim

MANY add
ZERO mult

MANY add
1 mult

MANY add
MANY mult

… and some bold attempts [Fellows-Koblitz] …

… which were quickly broken …

… until, in October 2008 …

… until, in October 2008 …

… *Craig Gentry* came up with the first

fully homomorphic encryption scheme …

How does it work?

What is the magic?

Gentry’s scheme was complex …

… it used advanced algebraic number theory …

Some of us asked: can we make this really simple? …

Some of us asked: can we make this really simple? …

Polynomials?

$(x^2 + 6x + 1)$ + $(x^2 - 6x) = (2x^2 + 1)$

$(x^2 + 6x + 1)$ X $(x^2 - 6x) = (x^4 - 35x^2 - 6x)$

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Matrices?

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} X \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 3 \end{pmatrix}$$

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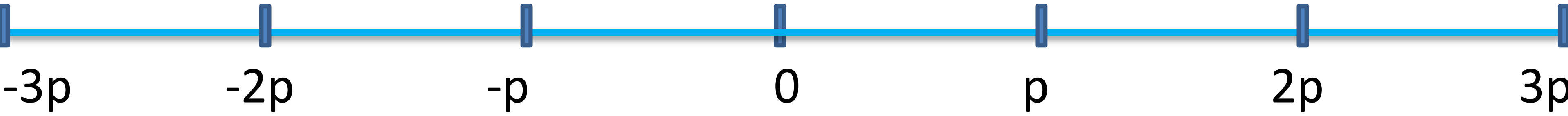
How about ***integers?!?*** [Gentry, Halevi, van Dijk, **V.**]

$$2\ +\ 3=5$$

$$2\ \text{X}\ 3=6$$

TODAY: Secret-key (Symmetric-key) Encryption

Secret key: large *odd* number **p**



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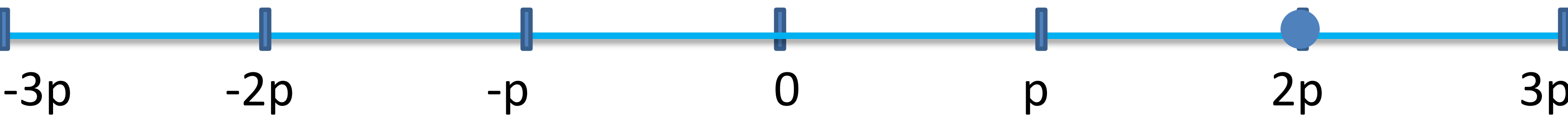
*To Encrypt a bit **b**:*

– pick a (random) “large” multiple of p, say **q·p**



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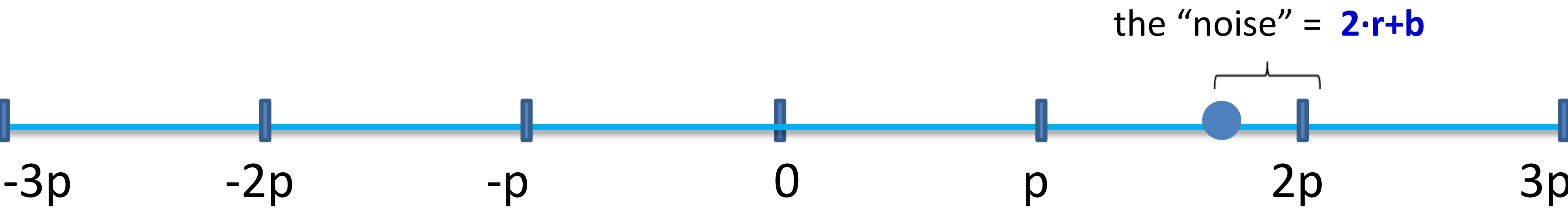
(this is even if b=0, and odd if b=1)



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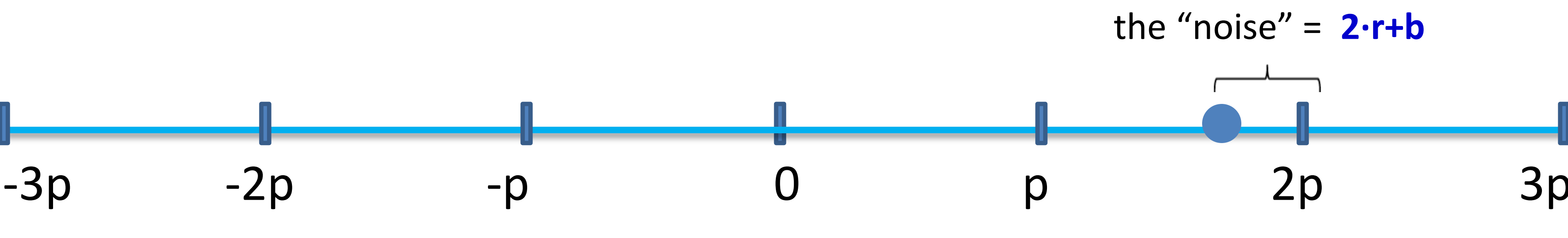

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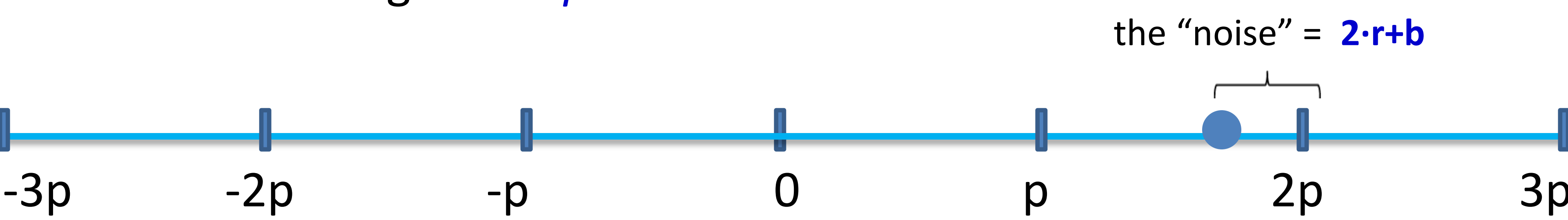
*To Decrypt a ciphertext **c**:*

Taking ***c*** *mod p* recovers the noise



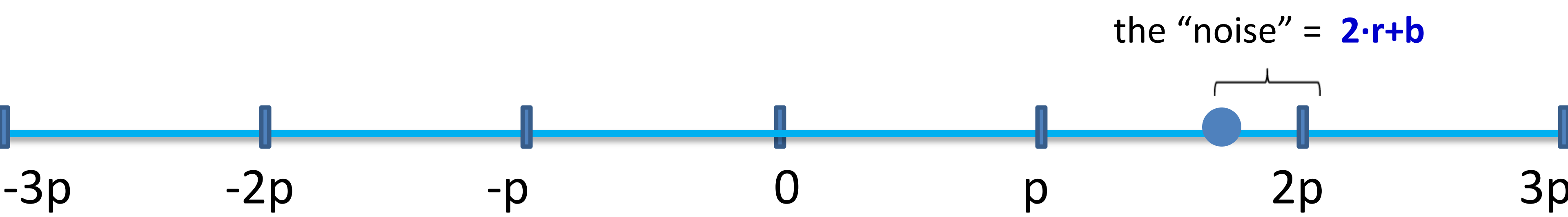
How secure is this?

... if there were no noise (think $r=0$)

... and I give you two encryptions of 0 (q_1p & q_2p)

... then you can recover the secret key p

$$= \mathrm{GCD}(q_1p,\ q_2p)$$



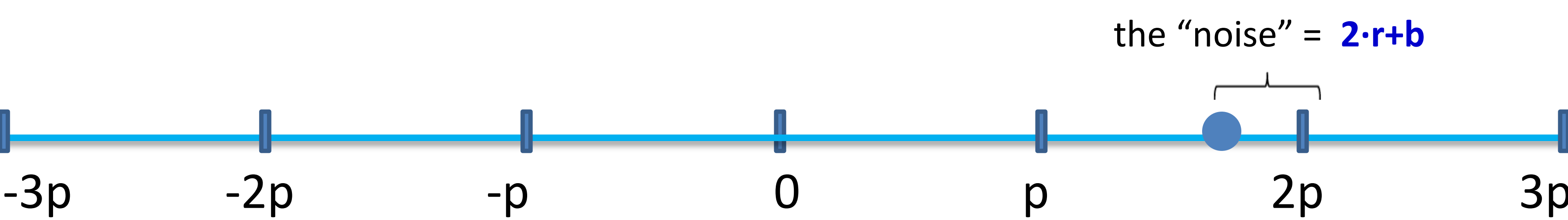
How secure is this?

… but if there is noise

… the GCD attack doesn't work

… and neither does any attack (we believe)

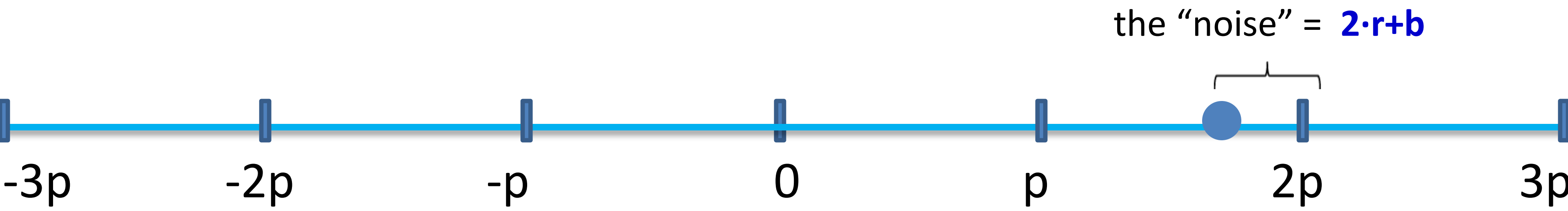
… this is called the *approximate GCD assumption*



XORing two encrypted bits:

- $\mathbf{c_1} = q_1 \cdot p + (2 \cdot r_1 + b_1)$

- $\mathbf{c_2} = q_2 \cdot p + (2 \cdot r_2 + b_2)$



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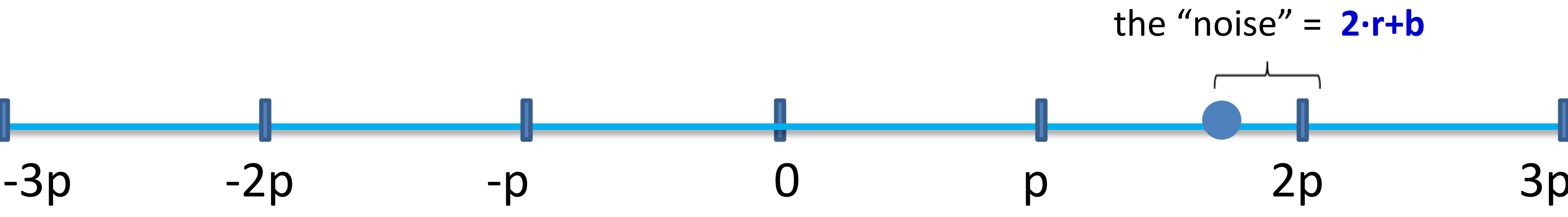

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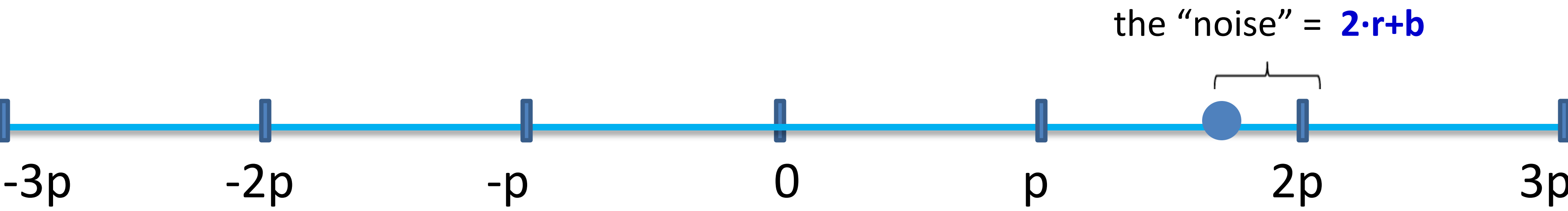
Odd if $b_1=0, b_2=1$ (or)
$b_1=1, b_2=0$

Even if $b_1=0, b_2=0$ (or)
$b_1=1, b_2=1$



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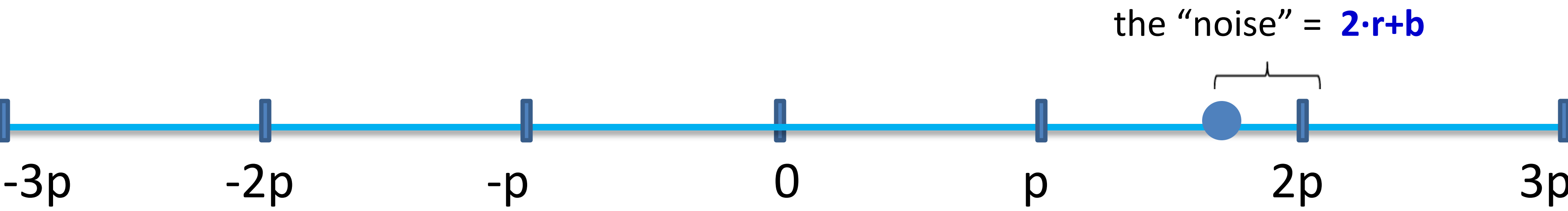

ANDing two encrypted bits:

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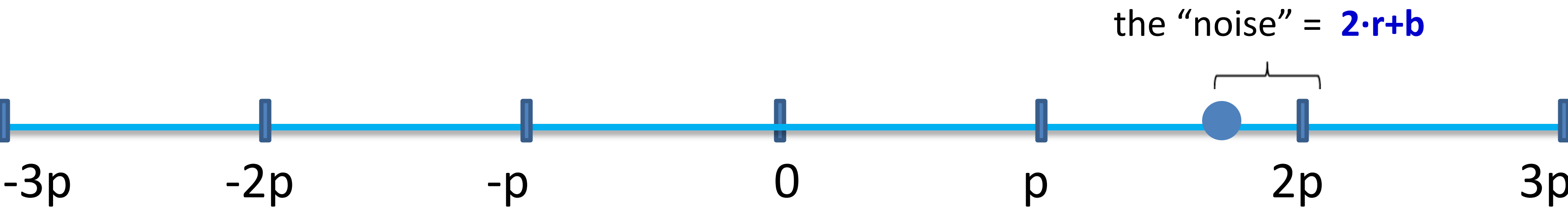
- $\mathbf{c_2} = q_2 \cdot p + (2 \cdot r_2 + b_2)$

- $\mathbf{c_1 c_2} = \mathbf{p\cdot}(c_2 \cdot q_1 + c_1 \cdot q_2 - q_1 \cdot q_2) + \mathbf{2\cdot}(r_1 r_2 + r_1 b_2 + r_2 b_1) + b_1 b_2$



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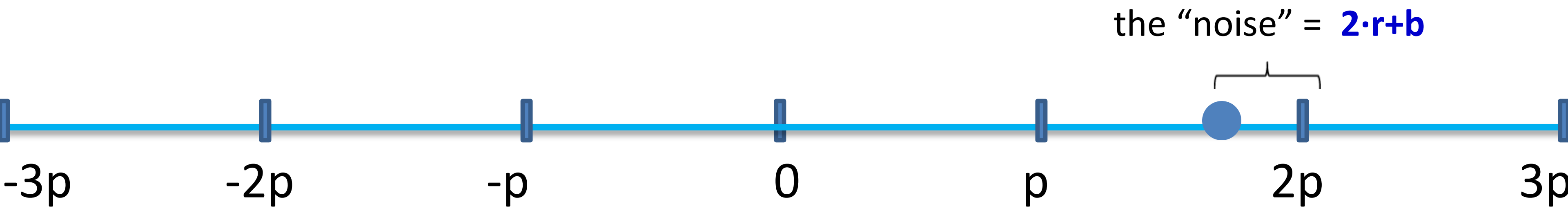
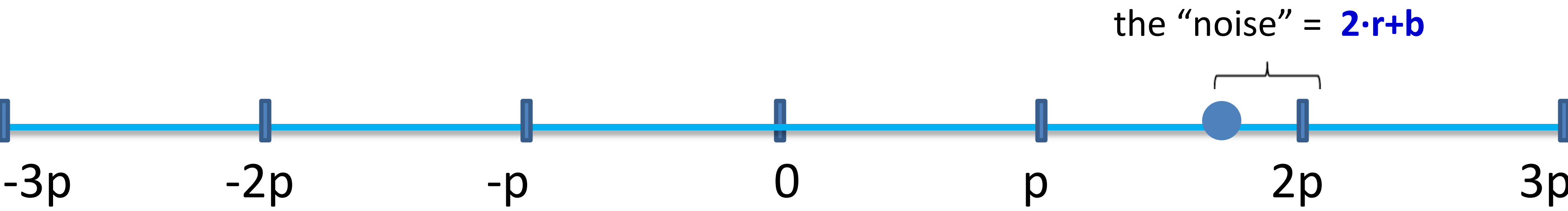
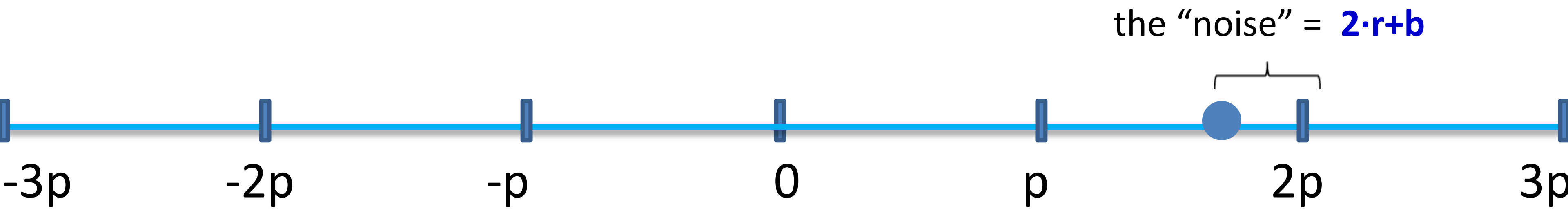


the noise grows!



the noise grows!

- $c_1+c_2 = p\cdot(q_1+q_2) + \underbrace{2\cdot(r_1+r_2) + (b_1+b_2)}_{\textit{noise= 2 * (initial noise)}}$



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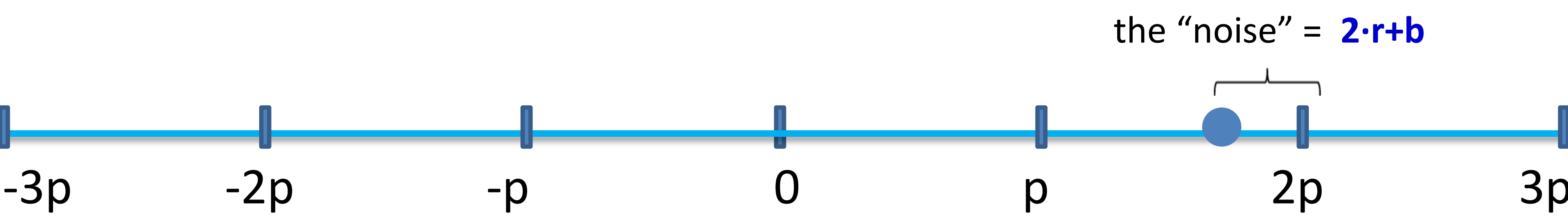
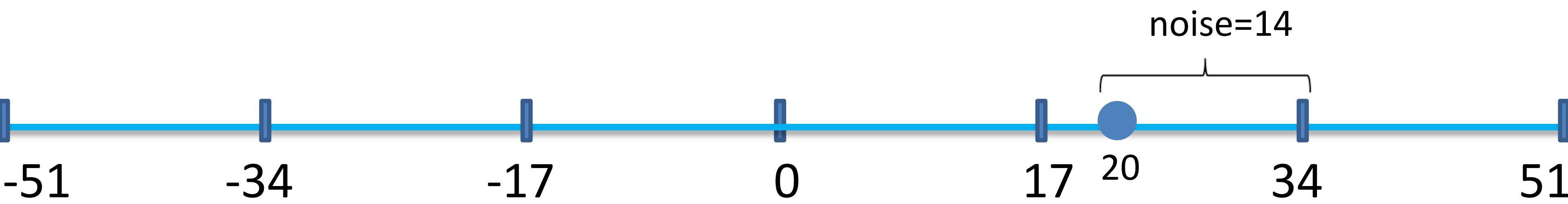


the noise grows!

… so what’s the problem?



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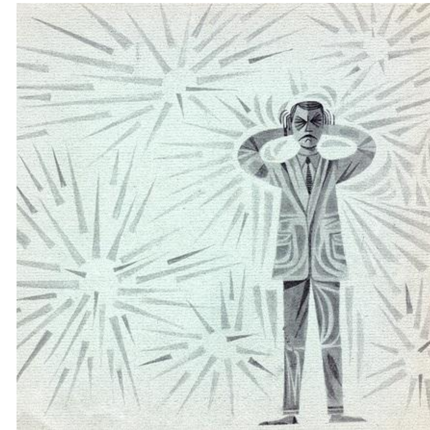

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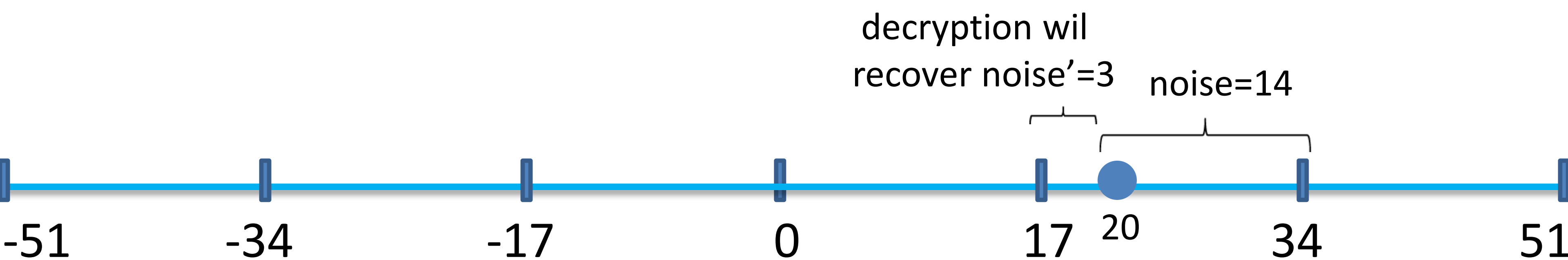
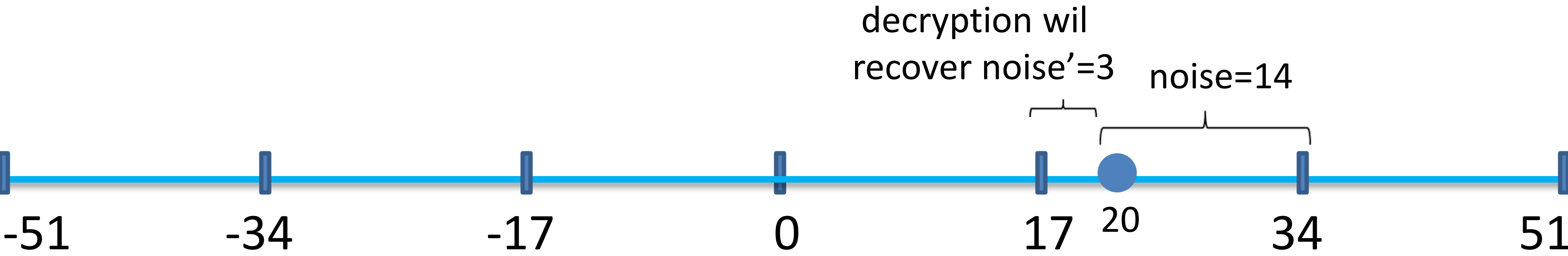


the noise grows!

… so what’s the problem?

If the |noise| > p/2, then …

decryption will output an incorrect bit



So, what did we accomplish?

… we can do lots of additions and

… some multiplications

(= a “somewhat homomorphic” encryption)

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… enough to do many useful tasks, e.g., database search, spam filtering etc.

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… some multiplications

(= a "somewhat homomorphic" encryption)

… enough to do many useful tasks, e.g.,
database search, spam filtering etc.

But I promised much more …

Josh's system
Boneh,Goh & Nissim
Fully homomorphic
MANY add
ZERO mult
MANY add
1 mult
WE ARE HERE!
MANY add
MANY mult

Gentry's ***"bootstrapping theorem" …***

Gentry's ***"bootstrapping theorem"*** ***…***

… If you can go a (large) part of the way,

then you can go all the way.

Gentry's ***"bootstrapping theorem"*** ...

... If you can go a (large) part of the way,

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[HOW? WE'LL SEE IN A BIT]

How efficient is all this?

… can I buy a homomorphic encryption software and start encrypting my data?

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… well, not quite yet

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… well, not quite yet

… encrypting a bit takes ~19s (!) with the current best implementation

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… can I buy a homomorphic encryption software and start encrypting my data?

… well, not quite yet

… encrypting a bit takes ~19s (!) with the current best implementation

… it takes 99 min to encrypt this sentence

How efficient is all this?

… can I buy a homomorphic encryption software and start encrypting my data?

… well, not quite yet

… encrypting a bit takes ~19s (!) with the current best implementation

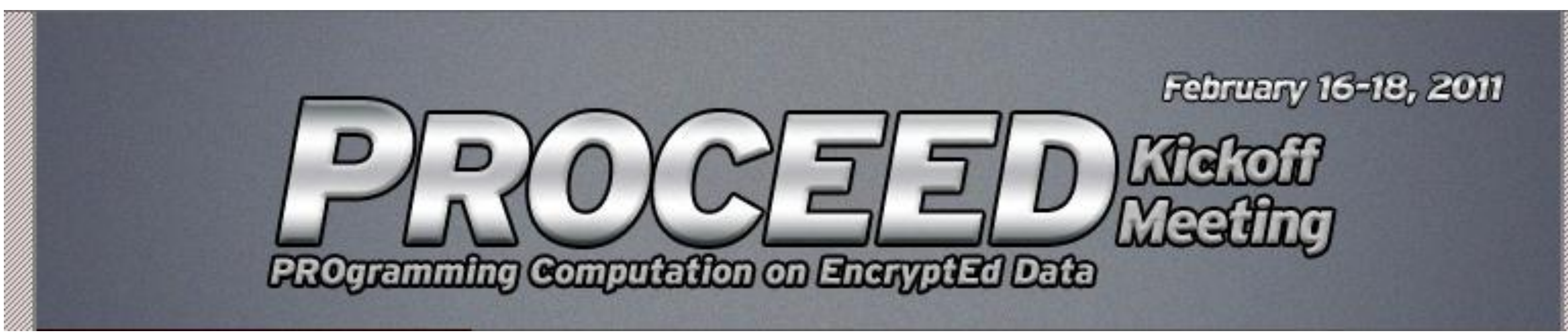
… but we are improving rapidly…

… a number of new, more efficient schemes

… optimized implementation efforts

(in hardware and software)

… and a $20M DARPA project to fund all this



… a number of new, more efficient schemes

… optimized implementation efforts

(in hardware and software)

… and a $20M DARPA project to fund all this

So, watch out for new developments!

References:

[1] "Computing arbitrary functions of Encrypted Data", Craig Gentry, Communications of the ACM 53(3), 2010.

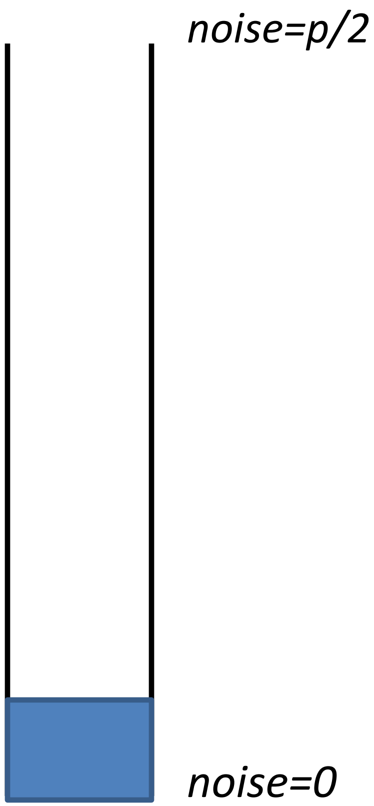
[2] "Fully Homomorphic Encryption from the Integers",
Marten van Dijk, Craig Gentry, Shai Halevi, Vinod Vaikuntanathan
http://eprint.iacr.org/2009/616, Eurocrypt 2010.

[3] "Implementing Gentry's Fully Homomorphic Encryption",
Craig Gentry and Shai Halevi
https://researcher.ibm.com/researcher/files/us-shaih/fhe-implementation.pdf, Eurocrypt 2011.

Gentry's ***"bootstrapping method" …***

… If you can go a (large) part of the way,

then you can go all the way…



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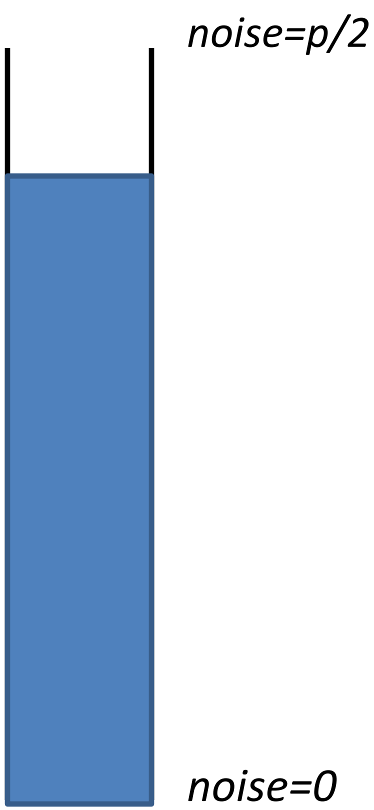

Problem: Add and Mult increase noise

(Add doubles, Mult squares the noise)

Gentry's ***"bootstrapping method" …***

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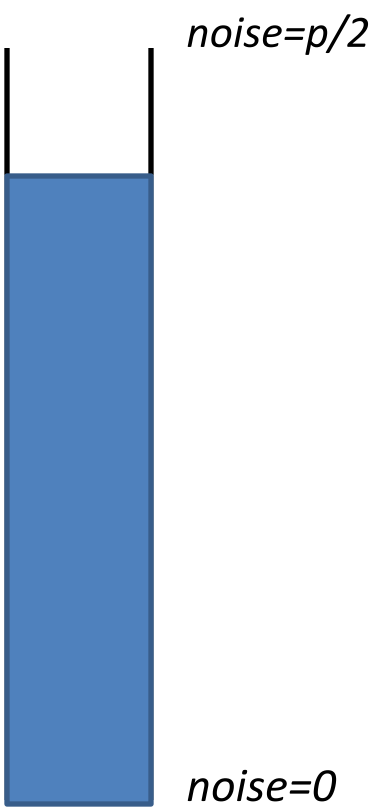


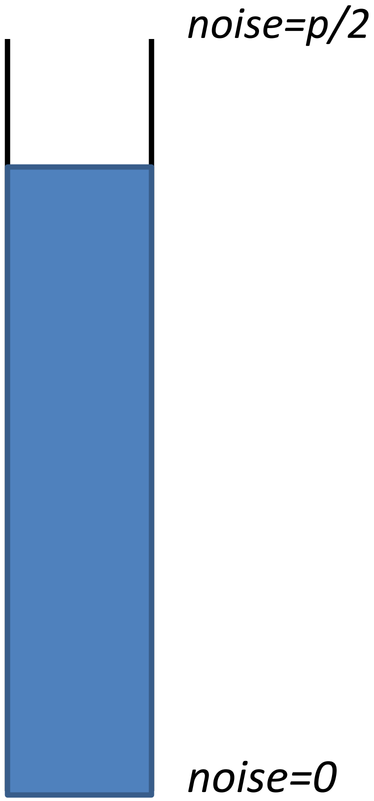
Problem: Add and Mult increase noise

(Add doubles, Mult squares the noise)

So, we want to do *noise-reduction*

Let's think…

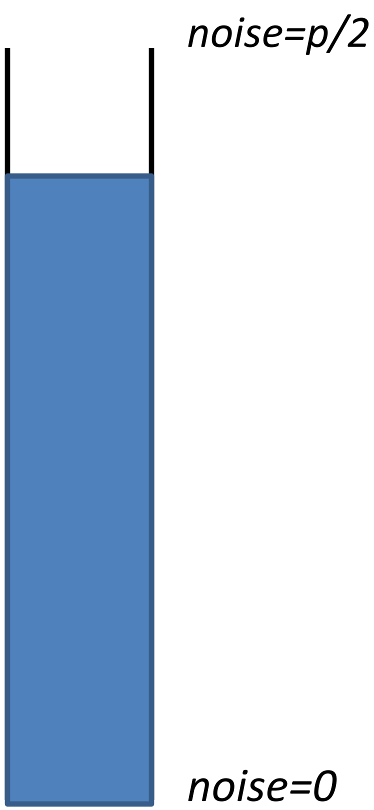
… What is the best noise-reduction procedure?



Let's think…

… What is the best noise-reduction procedure?

… something that kills all noise



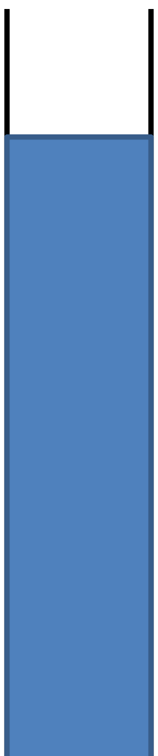
Let's think…

… What is the best noise-reduction procedure?

… something that kills all noise

noise=p/2

… and recovers the message



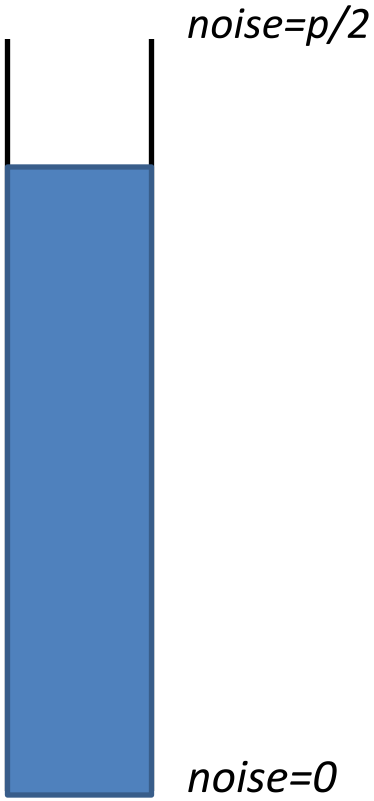
noise=0

Let's think…

… What is the best noise-reduction procedure?

… something that kills all noise

… and recovers the message



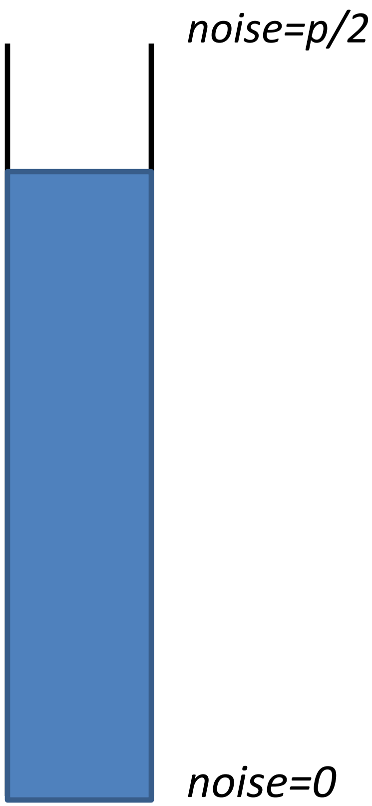
Decryption!

Let's think…

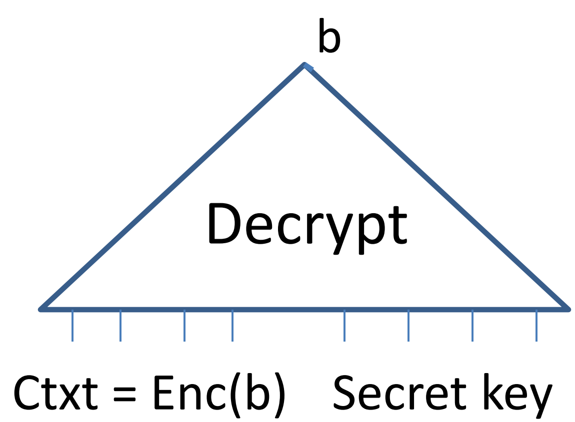
… What is the best noise-reduction procedure?

… something that kills all noise

… and recovers the message



Decryption!



Let's think…

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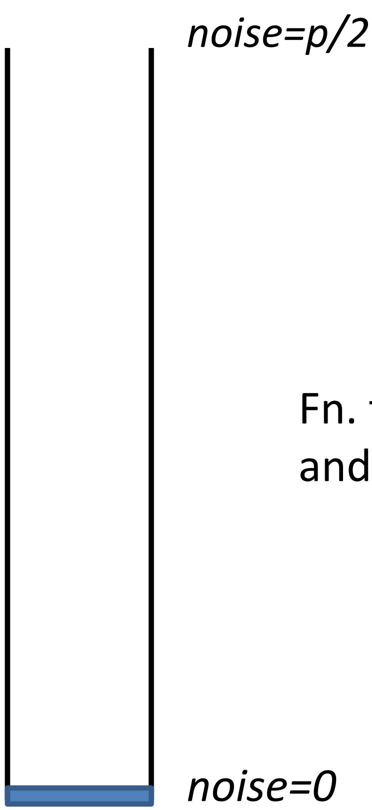
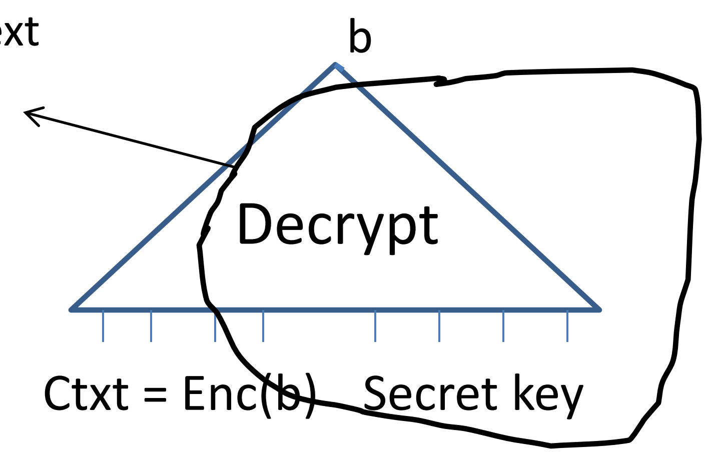


Decryption!

Fn. that acts on ciphertext
and eliminates noise



Let’s think…

… What is the best noise-reduction procedure?

… something that kills all noise

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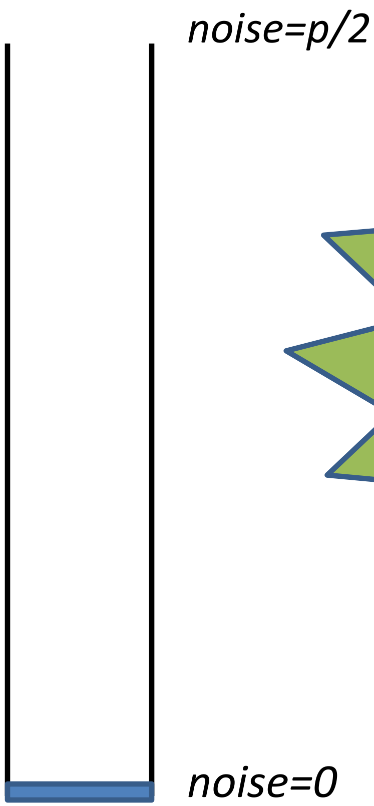
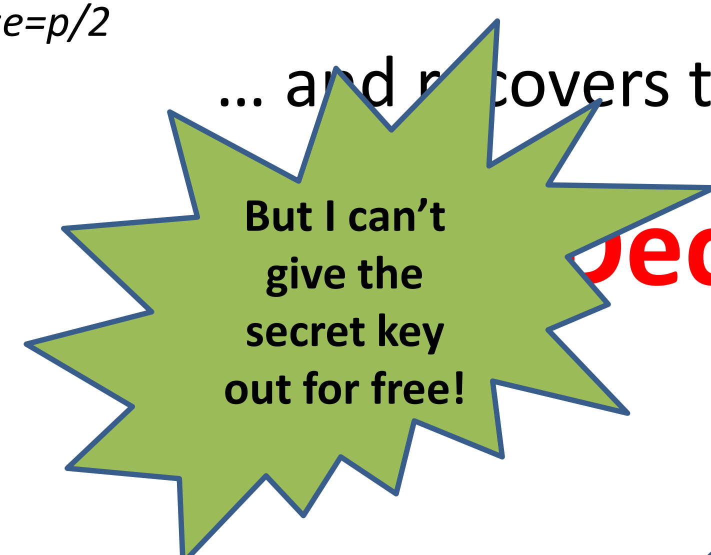
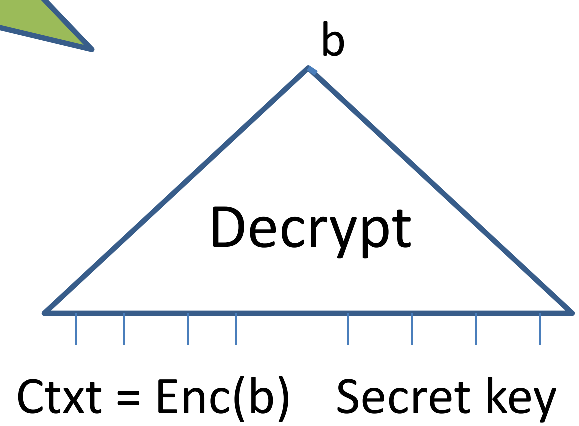
Decryption!

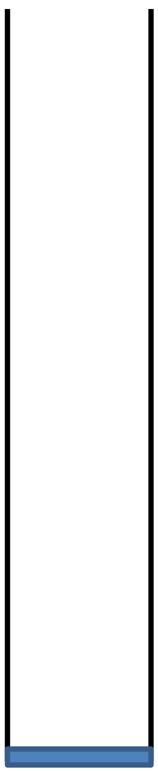




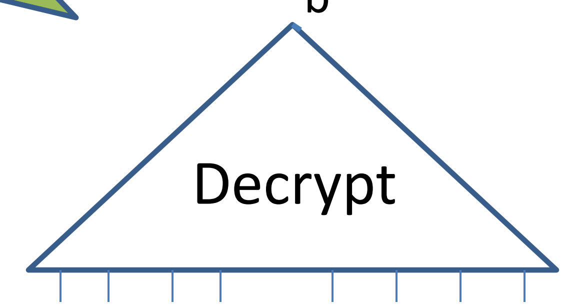


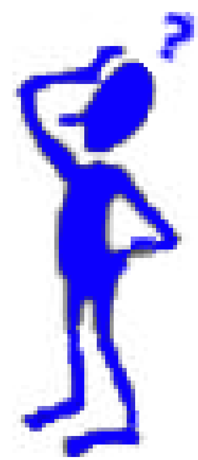
Let's think…

… I want to reduce noise *without letting you decrypt*



But I can't give the secret key out for free!

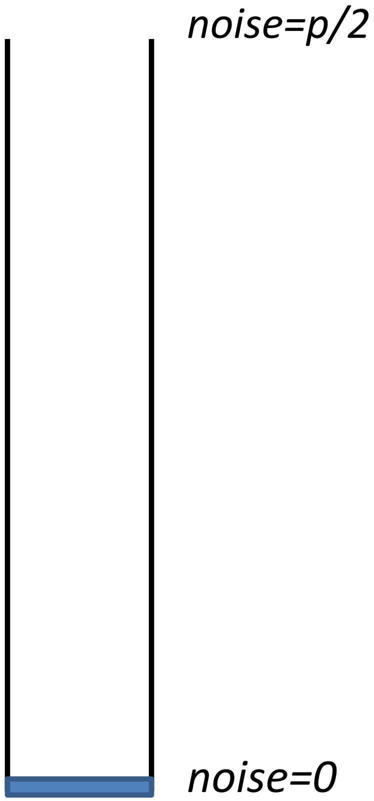


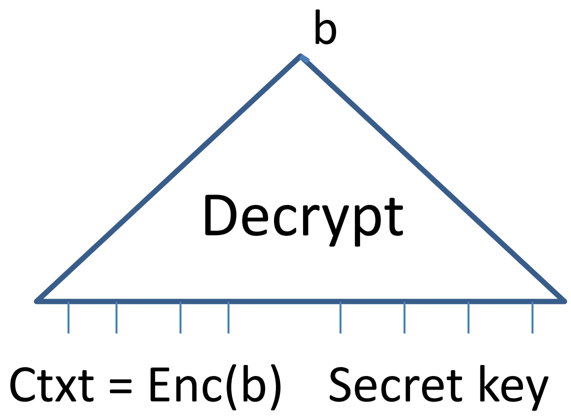


KEY IDEA:

… I cannot release the secret key (lest everyone sees my data)

… but *I can release Enc(secret key)*





KEY IDEA:

… I cannot release the secret key (lest everyone sees my data)

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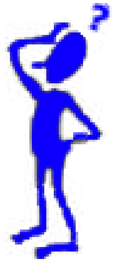
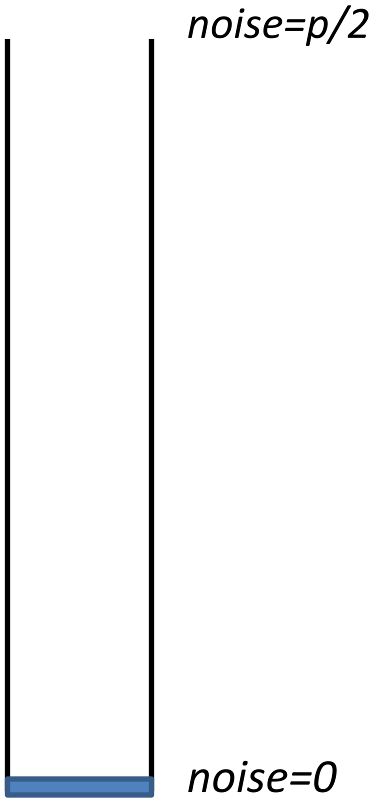
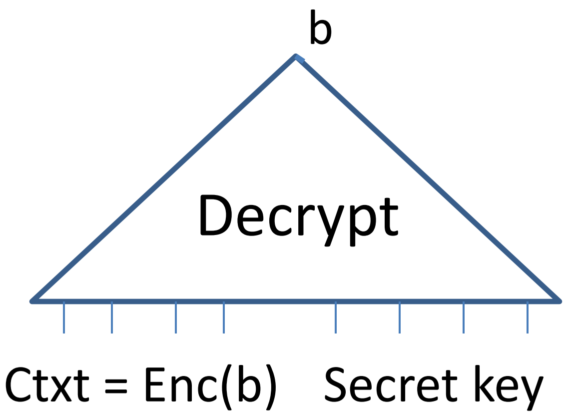
… called “Circular Encryption”







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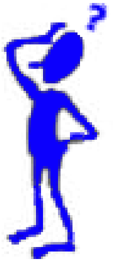

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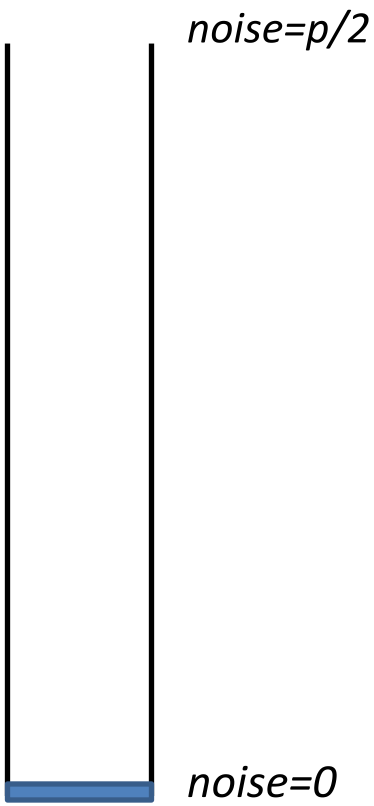

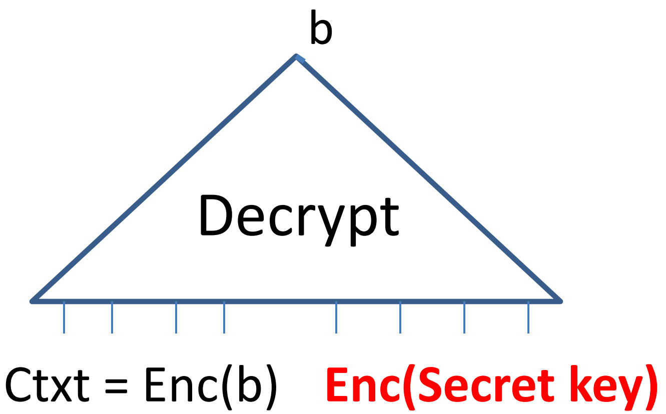

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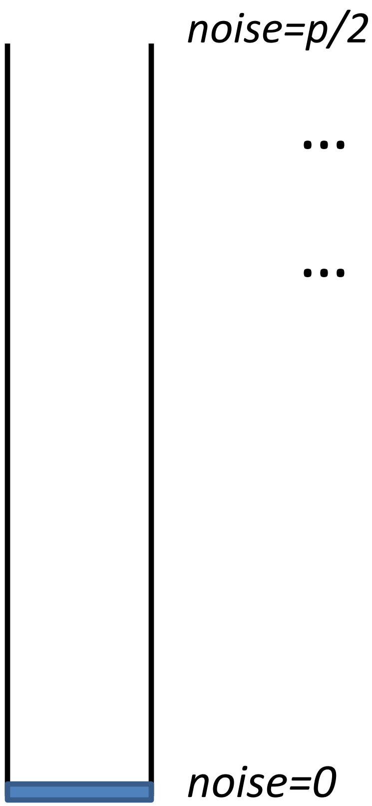
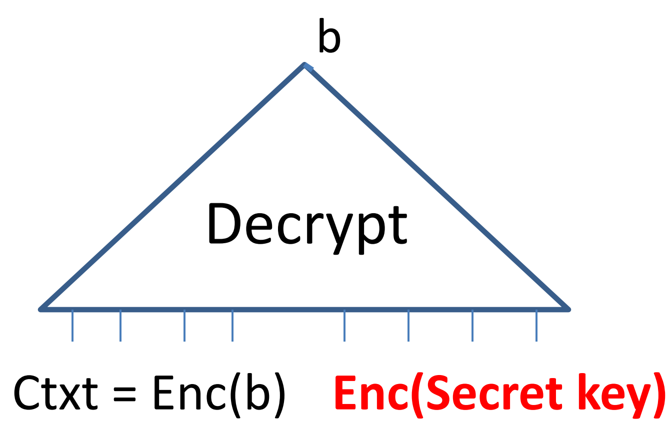


… Now, to reduce noise …

… Homomorphically evaluate the decryption ckt!!!



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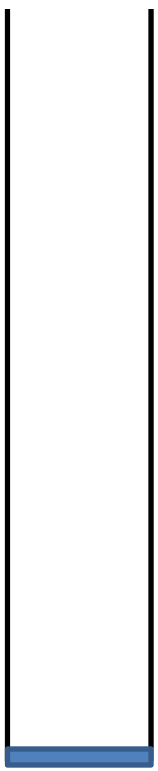
noise=p/2

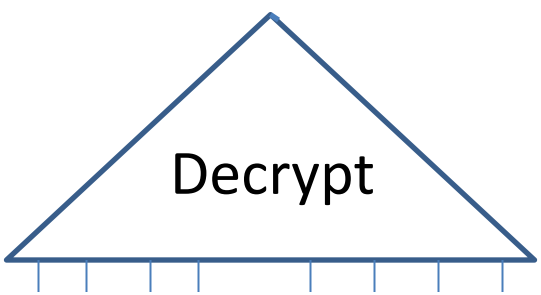


noise=0

... Now, to reduce noise ...

... Homomorphically evaluate the decryption ckt!!!

Enc(b)



Ctxt = Enc(b) **Enc(Secret key)**

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… I cannot release the secret key (lest everyone sees my data)

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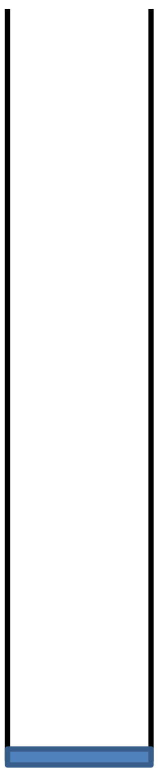

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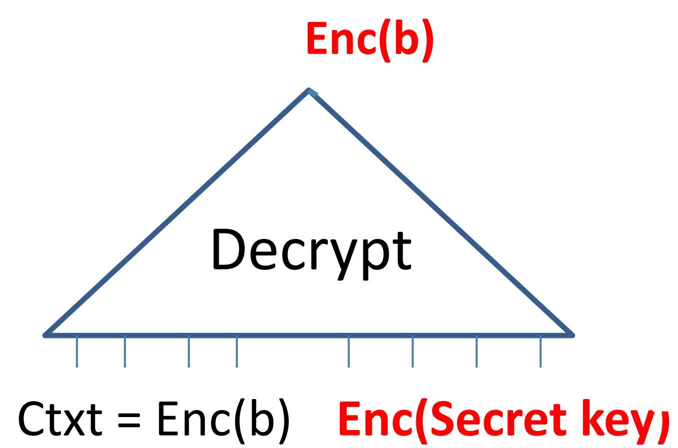

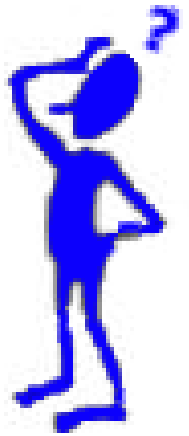

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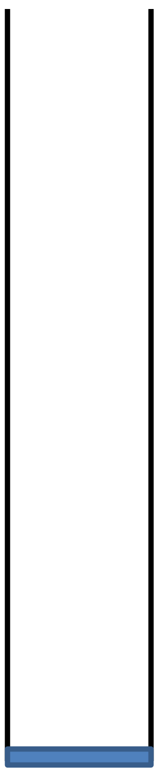
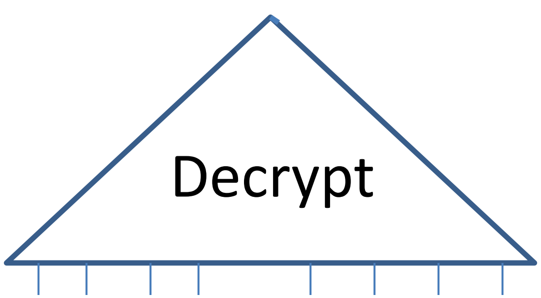


KEY OBSERVATION:

… the input Enc(b) and output Enc(b) have different noise levels …



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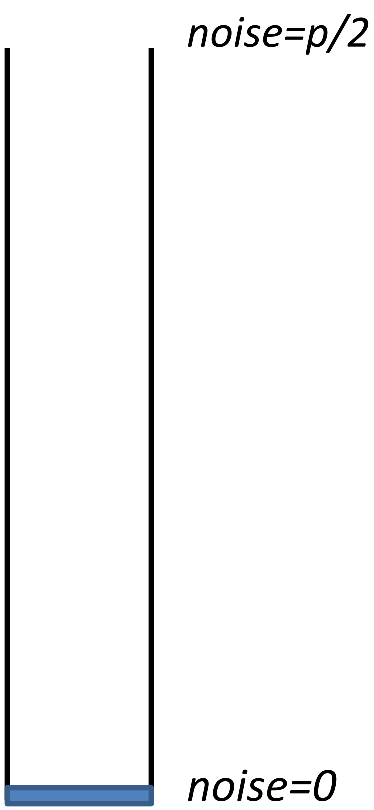
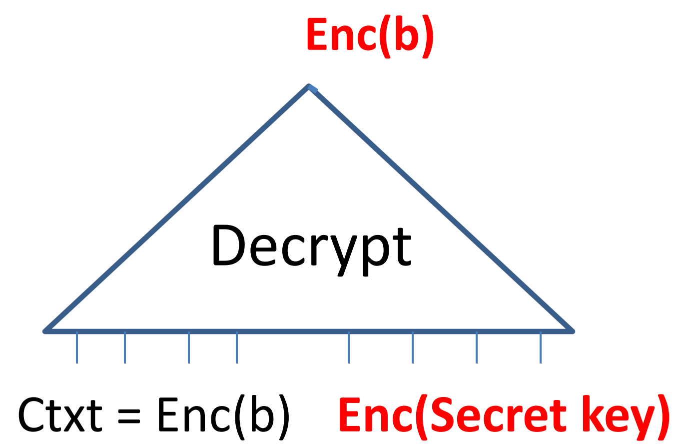


KEY OBSERVATION:

Regardless of the noise in the input Enc(b)…

the noise level in the output Enc(b) is **FIXED**



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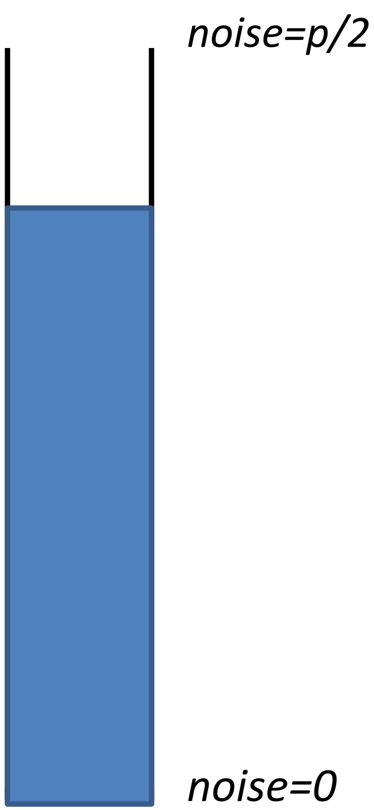

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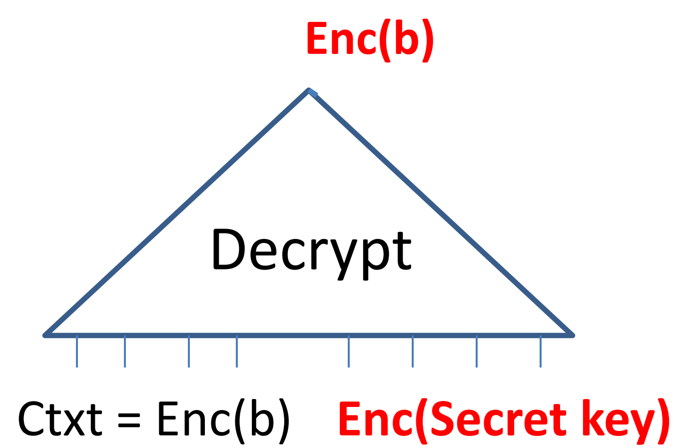

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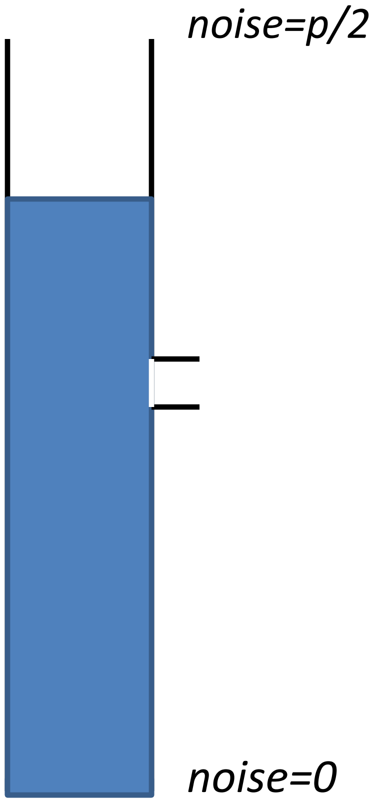

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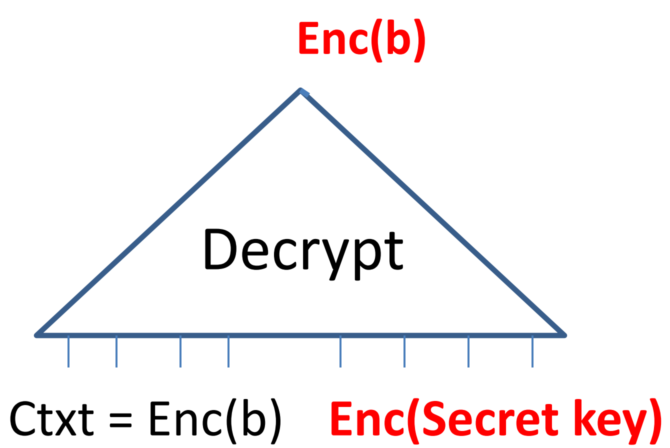

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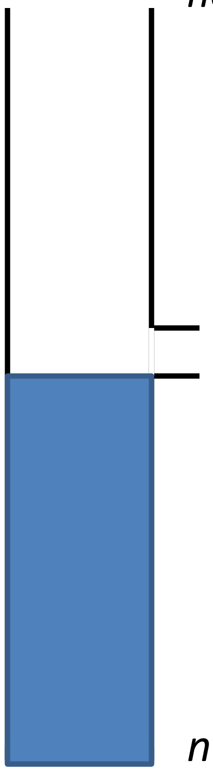

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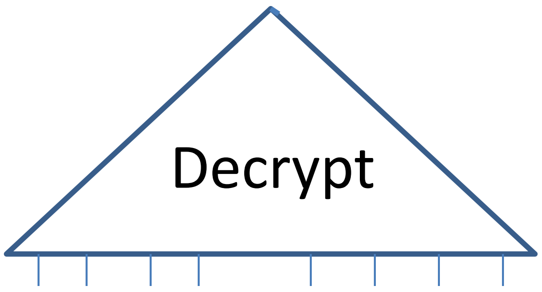
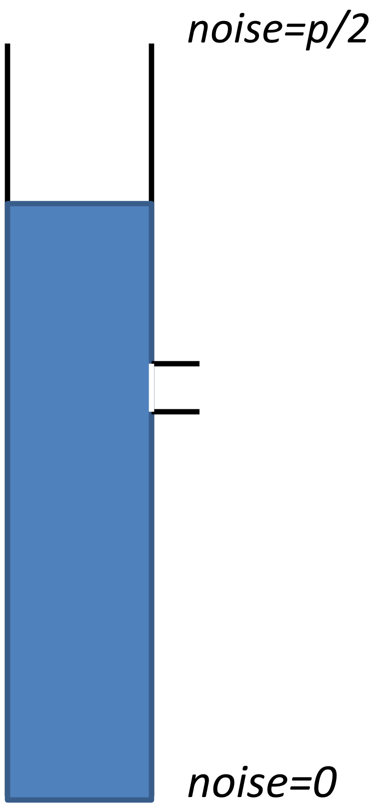
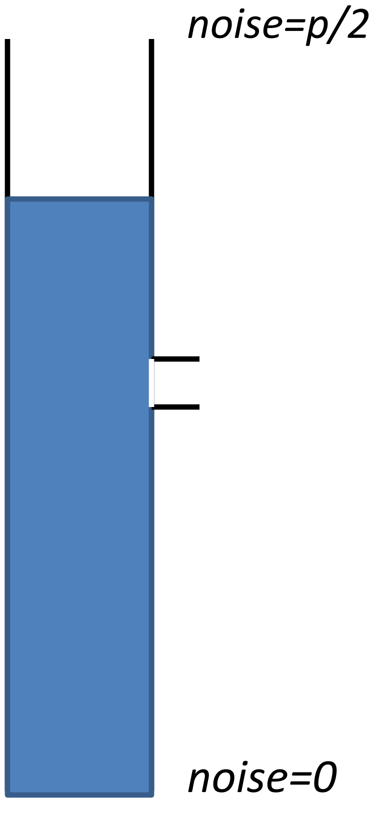


Bottomline: *whenever noise level increases beyond a limit …*

… use bootstrapping to reset it to a fixed level



Bootstrapping requires homomorphically evaluating the decryption circuit …



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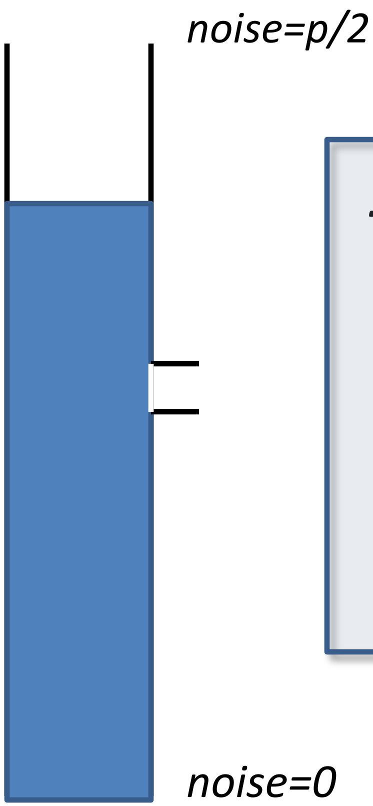

*Thus, Gentry's **"bootstrapping theorem":***

If an enc scheme can evaluate its own decryption circuit, then it can evaluate everything